Introduction to Related Rates

What has always distinguished calculus from algebra is its ability to deal with variables that change over time. It is quite easy to move from a formula relating static variables to a formula that relates their rates of change: Simply differentiate the formula implicitly **with respect to time**. This introduces an important category of problems called *related rates problems* that constitutes one of the most important applications of calculus.

1. Rewrite each statement mathematically.

a) The area of the rectangle is increasing at a rate of 4 square feet per second. $\frac{dA}{dt} = 4\ ft^2/sec$	b) The volume of water in the tank is decreasing at a rate of 10 cubic meters per minute. $\frac{dV}{dt} = -10\ m^3/min$	c) The height of the boy is 5 feet. $h = 5\ ft.$
d) The distance between the two cars is increasing at a rate of 40 mph. $\frac{dD}{dt} = 40\ mph$	e) The radius of the balloon is decreasing at a rate of 3 inches per second. $\frac{dr}{dt} = -3\ in/sec$	f) The height of the tree is increasing at a rate of 1 foot per year. $\frac{dh}{dt} = 1\ ft/yr$

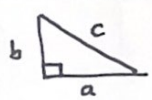
Many formulas contain variables that have a familiar relationship. In other words, a way that variables are "related" to each other. One great example is the Pythagorean Theorem.

$$a^2+b^2=c^2$$
$$x^2+y^2=z^2$$

Let's say that a right triangle has dimensions that are changing. ~~The~~ This means a dimension has a given rate of change. Since the dimensions are all related, then that also means that the rates of change of the dimensions are related.

Differentiate the relationship between the dimensions of a right triangle **with respect to time**. We will be using implicit differentiation.

$$a^2+b^2=c^2$$
$$2a\cdot\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

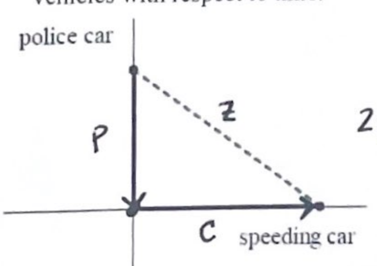


Implicit Differentiation

Recall: "with respect to x"

$$y^3=x$$
$$3y^2\frac{dy}{dx}=1$$

2. A police car, approaching a right-angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. Set up a relationship, then find an equation that shows related rates of the vehicles with respect to time.

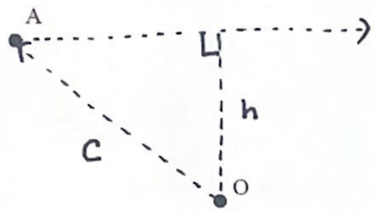


$$p^2+c^2=z^2$$
$$2p\frac{dp}{dt}+2c\frac{dc}{dt}=2z\frac{dz}{dt}$$

$$\frac{dp}{dt}<0 \quad \frac{dc}{dt}>0 \quad \boxed{\frac{dz}{dt}\overset{?}{<}0}$$

Some dimensions are constants

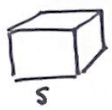
3. An airplane (point A) is flying on a horizontal path that will take it directly over an observer (point O). The airplane maintains a constant altitude. Relate the rates of change of the distance between the observer and the airplane and the horizontal distance between the two.



$$A^2 + H^2 = C^2$$

$$2A \cdot \frac{dA}{dt} + 2H\frac{dH}{dt} = 2C\frac{dC}{dt}$$

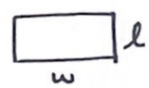
4. An ice cube is melting. Relate the rate of the change of the volume of the ice cube to the rate of change of the edges.



$$V = s^3$$

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$$

5. The width and length of a rectangle are increasing. Relate the rate of the change of the rectangle's area to the rates of change of the dimensions.



$$A = l \cdot w$$

$$\frac{dA}{dt} = l \cdot \frac{dw}{dt} + w\frac{dl}{dt}$$

6. A spherical balloon is expanding. Relate the rate of change of the surface area with the rate of change of the radius of the balloon. The surface area of a sphere is given by $S = 4\pi r^2$.

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

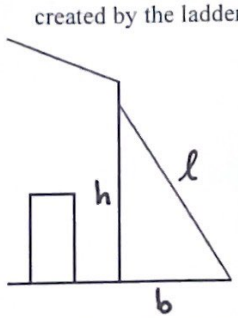
7. The water level in a cylindrical tank is dropping due to a small leak in the tank. Relate the rates of change between the water level and the volume of water. The volume of a cylinder is given by $V = \pi r^2 h$.

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$$

* r is constant

8. Mr. Werdel is using a ladder to paint his house. The ladder is leaning against the house when Mr. Q decides to pull the base of the ladder away from the house. Relate the rates of change for the dimensions of the triangle that is created by the ladder, the wall, and the ground.

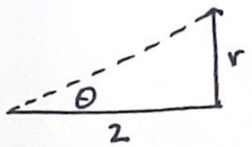


* l is constant

$$b^2 + h^2 = l^2$$

$$2b\frac{db}{dt} + 2h\frac{dh}{dt} = 0$$

9. A rocket is rising vertically. An observer on the ground is standing 2 miles from the rocket's launch point. As the observer watches the rocket, the angle of elevation is increasing. Relate the rates of the change of the angle of elevation with the speed of the rocket.



$\tan\theta = \frac{r}{2}$

$2\tan\theta = r \rightarrow 2\sec^2\theta \frac{d\theta}{dt} = \frac{dr}{dt}$

10. The water level in a **conical** tank is dropping due to a small leak in the tank. Relate the rates of change between the water level and the volume of water. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.

$V = \boxed{\frac{1}{3}\pi r^2}\boxed{h}$ (product rule)

$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$

11. If $y = 3x^4 + 6x$, find $\frac{dy}{dt}$ when $x = 1$ and $\frac{dx}{dt} = -3$.

$\frac{dy}{dt} = 12x^3 \frac{dx}{dt} + 6\frac{dx}{dt}$

$\frac{dy}{dt} = 12(-3) + 6(-3) = -54$

12. If $a^2 + b^2 = c^2$, find $\frac{da}{dt}$ when $a = 4$, $b = 3$, $\frac{db}{dt} = -1$, and $\frac{dc}{dt} = 12$.

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

$3^2 + 4^2 = c^2$

$c = 5$

$8\frac{da}{dt} - 6 = 120$

$8\frac{da}{dt} = 126$

$\boxed{\frac{da}{dt} = \frac{63}{4}}$

13. If $A = \frac{1}{2}bh$, find $\frac{dA}{dt}$ when $b = 7$, $h = 6$, $\frac{db}{dt} = 2$, and $\frac{dh}{dt} = -3$.

$\frac{dA}{dt} = \frac{1}{2}b\frac{dh}{dt} + \frac{1}{2}h\frac{db}{dt}$

$\frac{dA}{dt} = \frac{7}{2}(-3) + 3(2) = -\frac{9}{2}$